3.10) $S = \text{gen}\{x^2\}^{\perp}$

a) $(p,q) = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$S = \text{gen}\{x^2\}^{\perp} = \{p \in \mathbb{R}_2[x] : (p, x^2) = 0\}$

$p \in S$ si $p(x) = ax^2 + bx + c$, $a,b,c \in \mathbb{R}$.

y $0 = (p, x^2) = (ax^2+bx+c, x^2)$

$\Rightarrow (a-b+c).1 + \cancel{c.0} + (a+b+c).1 = 0$

$\rightarrow 2a + 2c = 0 \rightarrow a = -c$

en $p(x) \rightarrow p(x) = -cx^2 + bx + c = b.(x) + c.(-x^2+1)$

Por lo tanto una base ortogonal de S es:

$\boxed{B_S = \{x, -x^2+1\}}$ ya que son LI y generadores de S.

Siendo $S = \text{gen}\{x^2\}^{\perp} \rightarrow S^{\perp} = (\text{gen}\{x^2\}^{\perp})^{\perp} = \text{gen}\{x^2\}$

entonces $B_{S^\perp} = \{x^2\}$

Junto las dos bases:

$B_S \cup B_{S^\perp} = \{x, -x^2+1, x^2\}$

Como es un conj. LI y tiene la misma dim. que $\mathbb{R}_2[x]$

entonces forma una base de $\mathbb{R}_2[x] \rightarrow S \oplus S^{\perp} = \mathbb{R}_2[x]$.

Para descomponerlo:

Siendo $p(x) = a_0 + a_1 x + a_2 x^2$

$\rightarrow a_0 + a_1 x + a_2 x^2 = \underbrace{\alpha.(x) + \beta.(-x^2+1)}_{p_S} + \underbrace{\gamma.(x^2)}_{p_{S^\perp}}$

Ec.:

$$\begin{cases} \boxed{\beta = a_0} \\ \boxed{\alpha = a_1} \\ -\beta + \gamma = a_2 \rightarrow \boxed{\gamma = a_2 + a_0} \end{cases}$$

$\rightarrow \underbrace{a_0 + a_1 x + a_2 x^2}_{p} = \underbrace{a_1.(x) + a_0.(-x^2+1)}_{p_S} + \underbrace{(a_2+a_0)(x^2)}_{p_{S^\perp}}$

con $a_1(x) + a_0(-x^2+1) \in S$ y $(a_2+a_0)(x^2) \in S^\perp$

6) $(p,q) = p(0)q(0) + p'(0)q'(0) + \frac{1}{2} p''(0) q''(0)$

$S = \text{gen}\{x^2\}^\perp = \{p \in \mathbb{R}_2[x] : (p, x^2) = 0\}$

$p(x) = ax^2 + bx + c \rightarrow p(0) = c$

$p'(x) = 2ax + b \rightarrow p'(0) = b$

$p''(x) = 2a \rightarrow p''(0) = 2a$

$q'(x) = 2x$

$q''(x) = 2$

Como $0 = (p, x^2) = (ax^2 + bx + c, \underbrace{x^2}_{q(x)})$

$\rightarrow c.0 + b.0 + \frac{1}{2}.2a.2 = 0$

$\rightarrow 2a = 0 \rightarrow a = 0.$

en $p(x) \rightarrow p(x) = b(x) + c.(1)$

Por lo tanto una base ortogonal de S es:

$B_S = \{x, 1\}$ ya que son LI y generan S.

Siendo $S = \text{gen}\{x^2\}^{\perp} \rightarrow S^{\perp} = (\text{gen}\{x^2\})^{\perp} = \text{gen}\{x^2\}$

entonces $B_{S^\perp} = \{x^2\}$

Junto las bases:

$B_S \cup B_{S^\perp} = \underbrace{\{x, 1, x^2\}}_{LI} \rightarrow$ Base de $\mathbb{R}_2[x]$.

~~[illegible]~~

$ax^2 + bx + c = \underbrace{\alpha.(x) + \beta.(1)}_{P_S} + \underbrace{\gamma.(x^2)}_{P_{S^\perp}}$

Ec:

$$\begin{cases} \boxed{\gamma = a} \\ \boxed{\alpha = b} \\ \boxed{\beta = c} \end{cases}$$

~~[illegible]~~

~~[illegible]~~

$\rightarrow \underbrace{ax^2 + bx + c}_{P} = \underbrace{bx + c}_{P_S} + \underbrace{ax^2}_{P_{S^\perp}}$

c) $(p,q) = \int_{-1}^{1} \frac{1}{2} p(x) q(x)\, dx$.

$S = gen\{x^2\}^{\perp} = \{p \in \mathbb{R}_2[x] : (p, x^2) = 0\}$

$p(x) = ax^2 + bx + c$

Como $(p, x^2) = 0 = (ax^2+bx+c,\ x^2)$

$\rightarrow \frac{1}{2} \int_{-1}^{1} (ax^2+bx+c) \cdot x^2 \rightarrow \frac{1}{2} \cdot \int_{-1}^{1} (ax^4 + bx^3 + cx^2)\, dx = 0 \rightarrow$

$\rightarrow \frac{1}{2} \cdot \left(\frac{a}{5} x^5 + \frac{b}{4} x^4 + \frac{c}{3} x^3 \right) \Big|_{-1}^{1} = 0 \rightarrow \frac{1}{2} \cdot \left[\left(\frac{a}{5} + \frac{b}{4} + \frac{c}{3} \right) - \left(-\frac{a}{5} + \frac{b}{4} - \frac{c}{3} \right) \right] = 0 \rightarrow$

$\rightarrow \frac{1}{2} \cdot \left(\frac{2}{5} a + \frac{2}{3} c \right) = 0 \rightarrow \frac{1}{5} a + \frac{1}{3} c = 0 \rightarrow a = \left(-\frac{1}{3} c \right) \cdot 5 \rightarrow a = -\frac{5}{3} c$

en $p(x) \rightarrow p(x) = -\frac{5}{3} c x^2 + bx + c = b \cdot (x) + c \cdot \left(-\frac{5}{3} x^2 + 1 \right)$

Por lo tanto una base ortogonal de S es:

$$\boxed{B_S = \left\{ x, -\frac{5}{3} x^2 + 1 \right\}}$$

Siendo $S = gen\{x^2\}^{\perp} \rightarrow S^{\perp} = \left(gen\{x^2\}^{\perp} \right)^{\perp} = gen\{x^2\}$.

$\rightarrow B_{S^{\perp}} = \{x^2\}$.

Junto las dos bases:

$B_S \cup B_{S^{\perp}} = \{ \underbrace{x, -\frac{5}{3} x^2 + 1, x^2}_{LI} \} \rightarrow$ Base de $\mathbb{R}_2[x]$

$\underbrace{ax^2 + bx + c}_{p} = \underbrace{\alpha \cdot (x) + \beta \cdot \left(-\frac{5}{3} x^2 + 1 \right)}_{p_S} + \underbrace{\gamma \cdot (x^2)}_{p_{S^{\perp}}}$

Ec:

$$\begin{cases} -\frac{5}{3}\beta + \gamma = a \rightarrow \boxed{\gamma = a + \frac{5}{3}\cdot c} \\ \boxed{\alpha = b} \\ \boxed{\beta = c} \end{cases}$$

$$\underbrace{ax^2+bx+c}_{P} = \underbrace{b.(x) + c.\left(-\frac{5}{3}x^2+1\right)}_{P_S} + \underbrace{\left(a+\frac{5}{3}\cdot c\right)x^2}_{P_{S^\perp}}$$

d) $(p,q) = \int_0^\infty p(x)q(x)\,e^{-x}\,dx.$

$S = \text{gen}\{x^2\}^\perp = \{p \in \mathbb{R}_2[x] : (p, x^2) = 0\}.$

$p(x) = ax^2 + bx + c.$

Como $(p, x^2) = 0 = (ax^2+bx+c, x^2)$.

$\rightarrow \int_0^\infty (ax^2+bx+c).\,x^2.\,e^{-x}\,dx = 0.$

$\rightarrow \int_0^\infty \underline{(ax^4+bx^3+cx^2)}\,\underline{e^{-x}}\,dx = 0$